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An inequality in character algebras

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Abstract

In this paper, we prove the following:

Theorem. *Let $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ denote a complex character algebra with $d \geqslant 2$ which is P-polynomial with respect to the ordering $A_0, A_1, \ldots, A_d$ of the distinguished basis. Assume that the structure constants p^h_{ij} are all nonnegative and the Krein parameters q^h_{ij} are all nonnegative. Let θ and θ' denote eigenvalues of A_1, other than the valency $k = k_1$. Then the structure constants $a_1 = p^1_{11}$ and $b_1 = p^1_{12}$ satisfy*

$$\left(\theta + \frac{k}{a_1+1}\right)\left(\theta' + \frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2}.$$

Let E and F denote the primitive idempotents of $\mathscr{A}$ associated with θ and θ', respectively. Equality holds in the above inequality if and only if the Schur product $E \circ F$ is a scalar multiple of a primitive idempotent of $\mathscr{A}$.

The above theorem extends some results of Jurišić, Koolen, Terwilliger, and the present author. These people previously showed the above theorem holds for those character algebras isomorphic to the Bose–Mesner algebra of a distance-regular graph.

Keywords: Character algebra; Bose–Mesner algebra; Distance-regular graph; Association scheme; Krein parameter

1. Introduction

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$ and valency k. Let θ and θ' denote eigenvalues of Γ other than k. In [3], Jurišić, Koolen and Terwilliger proved

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that the intersection numbers a_1 and b_1 satisfy

$$\left(\theta + \frac{k}{a_1+1}\right)\left(\theta' + \frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2}. \tag{1}$$

They called (1) the *fundamental bound*. They defined Γ to be *tight* whenever Γ is not bipartite and equality holds in (1). In [6], we interpret equality in (1) in terms of the primitive idempotents of Γ. Our main result in that paper is as follows.

Theorem 1.1 (Pascasio [6]). *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$ and valency k. Let θ and θ' denote the eigenvalues of Γ other than k, and let E and F denote the corresponding primitive idempotents. The following are equivalent*:

(i) *Equality holds in* (1).
(ii) *The entrywise product $E \circ F$ is a scalar multiple of a primitive idempotent of Γ.*

In this paper, we extend inequality (1) and Theorem 1.1 to the level of character algebras. The concept of a character algebra (or C-algebra for short) was introduced by Kawada in [4]. A C-algebra is a finite-dimensional, associative and commutative algebra over the complex field with a distinguished basis satisfying certain conditions (see formal definition in the next section). The notion of a character algebra generalizes the Bose–Mesner algebra of a commutative association scheme, the center of the group algebra of a finite group, and the algebra of complex-valued class functions on a finite group. Character algebras with nonnegative structure constants are equivalent to table algebras.

There exists an algebraic generalization of a distance-regular graph, known as a P-polynomial character algebra (a formal definition is given in Section 3). In this paper, we show that inequality (1) and an analog of Theorem 1.1 hold for P-polynomial character algebras with nonnegative structure constants and Krein parameters. To obtain our results, we follow a method established by Maclean in [5].

2. Preliminaries

In this section, we collect some results on character algebras that will be useful in proving our main theorem. The reader is referred to [2] for more details. In the following, $\mathbb{C}$ denotes the complex field.

Definition 2.1. By a character algebra (or C-algebra), we mean a finite-dimensional associative $\mathbb{C}$-algebra $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ together with a distinguished basis $A_0, A_1, \ldots, A_d$ having the following properties:

(i) $\mathscr{A}$ is commutative.
(ii) A_0 is the multiplicative identity element of $\mathscr{A}$.

(iii) Let $\{p^h_{ij} \mid 0 \leqslant h, i, j \leqslant d\}$ denote complex numbers such that

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \quad (0 \leqslant i, j \leqslant d). \tag{2}$$

Then p^h_{ij} is a real number for $0 \leqslant h, i, j \leqslant d$.

(iv) There exists a permutation $i \mapsto i'$ of the set $\{0, 1, \ldots, d\}$ such that

$$(i')' = i \quad (0 \leqslant i \leqslant d),$$

$$p^h_{ij} = p^{h'}_{i'j'} \quad (0 \leqslant i, j, h \leqslant d).$$

(v) There exist positive real numbers $k_0, k_1, \ldots, k_d$ such that

$$p^0_{ij} = \delta_{ji'} k_i \quad (0 \leqslant i, j \leqslant d), \tag{3}$$

where $\delta_{ji'}$ is the Kronecker delta.

(vi) The linear map $\sigma : \mathscr{A} \to \mathbb{C}$ such that $\sigma : A_i \mapsto k_i$ $(0 \leqslant i \leqslant d)$ is a $\mathbb{C}$-algebra homomorphism.

We refer to the scalars $\{k_i \mid 0 \leqslant i \leqslant d\}$ and $\{p^h_{ij} \mid 0 \leqslant h, i, j \leqslant d\}$ as the *valencies* and *structure constants* of $\mathscr{A}$, respectively. For convenience, we set $n := \sum_{i=0}^{d} k_i$, and observe n is a positive real number.

Let $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ denote a C-algebra. By [2, pp. 89–93], there exist nonzero elements $E_0, E_1, \ldots, E_d$ of $\mathscr{A}$ such that

$$\begin{aligned} &E_0 = n^{-1}(A_0 + A_1 + \cdots + A_d), \\ &E_i E_j = \delta_{ij} E_i \quad (0 \leqslant i, j \leqslant d), \\ &E_0 + E_1 + \cdots + E_d = A_0. \end{aligned} \tag{4}$$

The $E_1, \ldots, E_d$ are unique up to permutation, and $E_0, E_1, \ldots, E_d$ form a basis for $\mathscr{A}$. We refer to $E_0, E_1, \ldots, E_d$ as the *primitive idempotents* of $\mathscr{A}$.

Let $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ denote a C-algebra and let $E_0, E_1, \ldots, E_d$ denote the primitive idempotents of $\mathscr{A}$. Since $E_0, E_1, \ldots, E_d$ form a basis for $\mathscr{A}$, there exist complex numbers $p_i(j)$ $(0 \leqslant i, j \leqslant d)$ such that

$$A_i = \sum_{j=0}^{d} p_i(j) E_j \quad (0 \leqslant i \leqslant d). \tag{5}$$

It follows from (4) and (5) that $A_i E_j = p_i(j) E_j$ for $0 \leqslant i, j \leqslant d$. The scalar $p_i(j)$ is called the *eigenvalue of A_i associated with E_j*. It is well known that $p_0(j) = 1$ $(0 \leqslant j \leqslant d)$,

$$p_i(0) = k_i \quad (0 \leqslant i \leqslant d) \tag{6}$$

and $\overline{p_i(j)} = p_{i'}(j)$ $(0 \leqslant i, j \leqslant d)$, where $-$ denotes complex conjugation. Assume for the moment that the structure constants of $\mathscr{A}$ are all nonnegative. Then by [1, Lemma 2.6],

$$|p_i(j)| \leqslant k_i \quad (0 \leqslant i, j \leqslant d). \tag{7}$$

Since $A_0, A_1, \ldots, A_d$ form a basis for $\mathscr{A}$, there exist complex numbers $q_i(j)$ $(0 \leqslant i, j \leqslant d)$ such that

$$E_i = n^{-1} \sum_{j=0}^{d} q_i(j) A_j \quad (0 \leqslant i \leqslant d). \tag{8}$$

For each integer j $(0 \leqslant j \leqslant d)$, set $m_j := q_j(0)$. It is well known that m_j is real and positive for $0 \leqslant j \leqslant d$, [2, p. 95]. Moreover, $q_0(i) = 1$ $(0 \leqslant i \leqslant d)$ and

$$\overline{p_i(j)} k_i^{-1} = q_j(i) m_j^{-1} \quad (0 \leqslant i, j \leqslant d). \tag{9}$$

Let $\circ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ denote the binary operation on A that is linear in each argument, and satisfies $A_i \circ A_j = \delta_{ij} A_i$ $(0 \leqslant i, j \leqslant d)$. The operation $\circ$ is known as the *Schur product*. Since $E_0, E_1, \ldots, E_d$ is a basis for $\mathscr{A}$, there exist complex numbers q_{ij}^h such that

$$E_i \circ E_j = n^{-1} \sum_{h=0}^{d} q_{ij}^h E_h \quad (0 \leqslant i, j \leqslant d). \tag{10}$$

It is known that q_{ij}^h is a real number for $0 \leqslant h, i, j \leqslant d$, [2, p. 100]. The q_{ij}^h are known as the *Krein parameters* of $\mathscr{A}$.

Lemma 2.2. *Let $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ denote a C-algebra, and let $E_0, E_1, \ldots, E_d$ denote the primitive idempotents of $\mathscr{A}$. Then* (i), (ii) *hold below for $0 \leqslant i, j \leqslant d$.*

(i) $E_i \circ E_j \neq 0$,
(ii) *at least one of $q_{ij}^0, q_{ij}^1, \ldots, q_{ij}^d$ is nonzero.*

Proof. (i) Using (8), write $E_i \circ E_j$ as a linear combination of $A_0, A_1, \ldots, A_d$, and observe that the coefficient of A_0 is not zero.

(ii) Immediate from (i) above and (10). □

Referring to part (ii) of the above Lemma, we now consider the case when exactly one of $q_{ij}^0, q_{ij}^1, \ldots, q_{ij}^d$ is nonzero. To understand this case, it is helpful to keep in mind the following result:

Lemma 2.3. *Let $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ denote a C-algebra, and let $E_0, E_1, \ldots, E_d$ denote the primitive idempotents of $\mathscr{A}$. Fix integers i, j, t $(0 \leqslant i, j, t \leqslant d)$. The following are equivalent*:

(i) *$E_i \circ E_j$ is a scalar multiple of E_t.*
(ii) *$q_{ij}^h = 0$ for all $h \in \{0, 1, \ldots, d\} \backslash t$.*

Proof. Follows from (10) and the linear independence of $E_0, E_1, \ldots, E_d$. □

3. *P*-polynomial character algebras

Definition 3.1. A C-algebra $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ is said to be P-polynomial whenever for all integers h,i,j $(0\leqslant h,i,j\leqslant d)$,

$$p_{ij}^h=0 \quad \text{if one of } h,i,j \text{ is greater than the sum of the other two,}$$

$$p_{ij}^h\neq 0 \quad \text{if one of } h,i,j \text{ is equal to the sum of the other two.} \tag{11}$$

Let $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ denote a P-polynomial C-algebra. Setting $h=0$ in (11), and using Definition 2.1(v), we find $i'=i$ for $0\leqslant i\leqslant d$. Abbreviate $c_i:=p_{1i-1}^i$ $(1\leqslant i\leqslant d)$, $a_i:=p_{1i}^i$ $(0\leqslant i\leqslant d)$, $b_i:=p_{1i+1}^i$ $(0\leqslant i\leqslant d-1)$, and define $b_d=c_0=0$. By (11), $c_i\neq 0$ $(1\leqslant i\leqslant d)$ and $b_i\neq 0$ $(0\leqslant i\leqslant d-1)$. Observe that $a_0=0$ and $c_1=1$ in view of (2). Observe by (3) that $k_1=b_0$; we denote this common value by k, and refer to it as the *valency* of $\mathscr{A}$. For convenience, we abbreviate $A:=A_1$.

By [2, p. 97], we find $a_i+b_i+c_i=k$ for $0\leqslant i\leqslant d$. Moreover, it follows from [2, Chapter II, Proposition 5.1] that

$$k_i=\frac{b_0b_1\ldots b_{i-1}}{c_1c_2\ldots c_i} \quad (0\leqslant i\leqslant d). \tag{12}$$

Setting $i=1$ in (2) we find

$$AA_i=c_{i+1}A_{i+1}+a_iA_i+b_{i-1}A_{i-1} \quad (0\leqslant i\leqslant d), \tag{13}$$

where $A_{-1}=A_{d+1}=0$.

In what follows, λ will denote an indeterminate, and $\mathbb{C}[\lambda]$ will denote the $\mathbb{C}$-algebra consisting of all polynomials in λ that have complex coefficients.

Definition 3.2. Let $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ denote a P-polynomial C-algebra with $d\geqslant 2$. Let $f_0,f_1,\ldots,f_d$ denote polynomials in $\mathbb{C}[\lambda]$ such that $f_0=1$ and

$$c_tf_{t-1}+a_tf_t+b_tf_{t+1}=\lambda f_t \quad (0\leqslant t\leqslant d-1), \tag{14}$$

where $f_{-1}=0$.

Example 3.3. Let $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ denote a P-polynomial C-algebra with $d\geqslant 2$. Then f_0,f_1,f_2 are given by

$$f_0=1, \qquad f_1=\frac{\lambda}{k}, \qquad f_2=\frac{\lambda^2-a_1\lambda-k}{kb_1}. \tag{15}$$

Lemma 3.4. *Let $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ denote a P-polynomial C-algebra with $d\geqslant 2$. Then for all integers i $(0\leqslant i\leqslant d)$,*

$$f_i(A)=\frac{A_i}{k_i}. \tag{16}$$

Proof. By (15), Eq. (16) holds for $i=0,1$. To prove Eq. (16) for $i>1$ we apply induction using Eqs. (14), (12) and (13). $\square$

Corollary 3.5. *Let* $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ *denote a* P*-polynomial* C*-algebra with* $d \geqslant 2$. *Then* A *generates* $\mathscr{A}$.

Let $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ denote a P-polynomial C-algebra. For convenience, we abbreviate $\theta_i := p_1(i)$ $(0 \leqslant i \leqslant d)$. Thus θ_i is the eigenvalue of A associated with E_i. We observe $\theta_0, \theta_1, \ldots, \theta_d$ are real and distinct since A generates $\mathscr{A}$. Observe by (6) that $\theta_0 = k$. If the structure constants of $\mathscr{A}$ are all nonnegative, then by (7) k is the maximum of $\theta_0, \theta_1, \ldots, \theta_d$.

Lemma 3.6. *Let* $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ *denote a* P*-polynomial* C*-algebra with* $d \geqslant 2$. *Let* $\theta_0, \theta_1, \ldots, \theta_d$ *denote the eigenvalues of* A. *Then for all integers* t $(0 \leqslant t \leqslant d)$,

$$q_j(t) = m_j f_t(\theta_j) \quad (0 \leqslant j \leqslant d).$$

Proof. Multiplying (5) and (16) by E_j, and comparing the resulting equations, we find

$$f_t(\theta_j) = p_t(j) k_t^{-1}.$$

The result now follows in view of (9), and the fact that $p_i(j)$ is real for $0 \leqslant i, j \leqslant d$. $\square$

Lemma 3.7. *Let* $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ *denote a* P*-polynomial* C*-algebra with* $d \geqslant 2$. *Let* $\theta_0, \theta_1, \ldots, \theta_d$ *denote the eigenvalues of* A. *Fix integers* i, j $(0 \leqslant i, j \leqslant d)$. *Then*

$$m_i^{-1} m_j^{-1} \sum_{h=0}^{d} q_{ij}^h m_h f_t(\theta_h) = f_t(\theta_i) f_t(\theta_j) \quad (0 \leqslant t \leqslant d).$$

Proof. Eliminating $E_0, E_1, \ldots, E_d$ in (10) using (8), and comparing the coefficients of A_t for $0 \leqslant t \leqslant d$, we find

$$q_i(t) q_j(t) = \sum_{h=0}^{d} q_{ij}^h q_h(t).$$

Evaluating this using Lemma 3.6, we get the result. $\square$

Lemma 3.8. *Let* $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ *denote a* P*-polynomial* C*-algebra with* $d \geqslant 2$. *Then*

$$1 = f_0, \qquad \lambda = k f_1, \qquad \lambda^2 = k b_1 f_2 + k a_1 f_1 + k f_0.$$

Proof. Solve the equations in (15) for $1, \lambda, \lambda^2$. $\square$

Lemma 3.9. *Let* $\mathscr{A} = \langle A_0, A_1, \ldots, A_d \rangle$ *denote a* P*-polynomial* C*-algebra with* $d \geqslant 2$. *Let* $\theta_0, \theta_1, \ldots, \theta_d$ *denote the eigenvalues of* A. *Fix integers* i, j $(0 \leqslant i, j \leqslant d)$. *Then*

(i) $m_i^{-1} m_j^{-1} \sum_{h=0}^{d} q_{ij}^h m_h = 1$
(ii) $m_i^{-1} m_j^{-1} \sum_{h=0}^{d} q_{ij}^h m_h \theta_h = \theta_i \theta_j / k$.

(iii) $m_i^{-1}m_j^{-1}\sum_{h=0}^{d} q_{ij}^h m_h\theta_h^2 = ((\theta_i - k)(\theta_j - k)/k^2b_1)((a_1+1)\theta_i\theta_j + k(\theta_i+\theta_j) + k(k-a_1)) + \theta_i^2\theta_j^2/k^2$.

Proof. Let us begin with line (iii). Expanding the left side of that line using Lemmas 3.8 (with $\lambda=\theta_h$) and 3.7, we obtain

$$kb_1f_2(\theta_i)f_2(\theta_j) + ka_1f_1(\theta_i)f_1(\theta_j) + kf_0(\theta_i)f_0(\theta_j).$$

Evaluating this using (15) we obtain the right side of line (iii). We have now proved (iii) of the present lemma. Lines (i), (ii) are similarly proved. □

4. Main result

Theorem 4.1. *Let $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ denote a P-polynomial C-algebra with $d\geqslant 2$. Assume that the structure constants and the Krein parameters of $\mathscr{A}$ are nonnegative. Let θ and θ' denote eigenvalues of A, other than k. Then*

$$\left(\theta+\frac{k}{a_1+1}\right)\left(\theta'+\frac{k}{a_1+1}\right) \geqslant -\frac{ka_1b_1}{(a_1+1)^2}. \tag{17}$$

Proof. Let $k=\theta_0>\theta_1>\cdots>\theta_d$ denote the eigenvalues of A. Since $\theta\neq k$ and $\theta'\neq k$, there exist integers i,j $(1\leqslant i,j\leqslant d)$ such that $\theta=\theta_i$ and $\theta'=\theta_j$. Recall q_{ij}^h are real and nonnegative for $0\leqslant h\leqslant d$. Further recall $m_0,m_1,\ldots,m_d$ are real and positive. It follows there exist vectors $v_s\in\mathbb{R}^{d+1}$ $(s=0,1)$ with h coordinate

$$\theta_h^s\sqrt{\frac{q_{ij}^h m_h}{m_im_j}} \quad (0\leqslant h\leqslant d). \tag{18}$$

Let $\langle\,,\rangle$ denote the dot product in $\mathbb{R}^{d+1}$. By the Cauchy–Schwarz inequality

$$\langle v_0,v_1\rangle^2 \leqslant \langle v_0,v_0\rangle\langle v_1,v_1\rangle. \tag{19}$$

We now evaluate the terms in (19). Observe

$$\langle v_r,v_s\rangle = m_i^{-1}m_j^{-1}\sum_{h=0}^{d} q_{ij}^h m_h\theta_h^{r+s} \quad (0\leqslant r,s\leqslant 1). \tag{20}$$

Let Δ denote the right-hand side of (19) minus the left-hand side of (19), and observe that Δ is nonnegative. Evaluating Δ using (20) and Lemma 3.9, we find Δ equals

$$\frac{(\theta_i-k)(\theta_j-k)}{k^2b_1} \tag{21}$$

times

$$(a_1+1)\theta_i\theta_j + k(\theta_i+\theta_j) + k(k-a_1). \tag{22}$$

Observe that (21) is positive so (22) is nonnegative. Recall a_1+1 is positive, and observe that (22) equals a_1+1 times

$$\left(\theta_i+\frac{k}{a_1+1}\right)\left(\theta_j+\frac{k}{a_1+1}\right)+\frac{ka_1b_1}{(a_1+1)^2}. \tag{23}$$

Apparently (23) is nonnegative, and the result follows. □

Concerning the case of equality in (17), we have the following result:

Corollary 4.2. *Let $\mathscr{A}=\langle A_0,A_1,\ldots,A_d\rangle$ denote a P-polynomial C-algebra with $d\geqslant 2$. Assume that the structure constants and the Krein parameters of $\mathscr{A}$ are nonnegative. Let θ and θ' denote eigenvalues of A, other than k. Let E and F denote the primitive idempotents corresponding to θ and θ', respectively. Then the following are equivalent*:

(i) *Equality holds in* (17).
(ii) *The Schur product $E\circ F$ is a scalar multiple of a primitive idempotent of $\mathscr{A}$.*

Proof. We refer to the proof of Theorem 4.1. (i)$\Rightarrow$(ii): We assume equality in (17), so (23) equals 0. Now (22) equals 0, so $\Delta=0$. We now have equality in (19), so v_0 and v_1 are linearly dependent. Let α,β denote real scalars not both zero, such that

$$\alpha v_0+\beta v_1=0. \tag{24}$$

The coordinates of v_0,v_1 are given on line (18). Evaluating (24) using this, we obtain $q_{ij}^h(\alpha+\beta\theta_h)=0$ for $0\leqslant h\leqslant d$. Observe that the second factor is zero for at most one h. By this and Lemma 2.2(ii), we find $q_{ij}^h\neq 0$ for exactly one h. Applying Lemma 2.3 we find $E_i\circ E_j$ is a scalar multiple of a primitive idempotent of $\mathscr{A}$.

(ii)$\Rightarrow$(i): By assumption $E_i\circ E_j$ is a scalar multiple of some primitive idempotent of $\mathscr{A}$. Denoting this by E_t, and applying Lemma 2.3, we find $q_{ij}^h=0$ for all $h\in\{0,1,\ldots,d\}\backslash t$.

We find that for the vectors v_0,v_1 all coordinates but t are zero. Apparently v_0,v_1 are linearly dependent. Now $\Delta=0$. It follows that (23) is zero, and we have equality in (17). □

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